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Power Counting and Regularization through Loop Integrations for Multiple Feynman Integrals in Minkowski Space[1]

Edward B. Manoukian

Department of National Defence, Royal Military College of Canada, Kingston, Ontario, K7L 2W3, Canada

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A regularization procedure with a regularization parameter δ is developed which may be applied to multiple Feynman integrals in Minkowski space. The regularization is carried out in *momentum* space and provides a rigorous method for studying Feynman integrals as multiple integrals in real variable theory. The regularized integrals are defined by changing the measure of integration $\prod_i dx_i$ to $\prod_i (1 + x_i^2)^{-\delta/2} dx_i$, $\delta > 0$, with a corresponding change defined in *Minkowski* space. We then develop a power counting convergence criterion for the absolute convergence of the integrals in terms of the parameter δ as a function of the so-called power asymptotic coefficients of Feynman integrands. An application to quantum electrodynamics is carried out.

1. INTRODUCTION

Regularization methods (e.g., Pauli and Villars, 1949; Slavnov, 1972) and dimensional regularization methods (e.g., 't Hooft and Veltman, 1972, Bollini and Giambiagi, 1972; Ashmore, 1972; Cicuta and Montaldi, 1972; Wilson, 1973; Blekher, 1982; Speer, 1974; DeVega and Schaposnik, 1974) have been quite useful in evaluating Feynman integrals which may be potentially divergent. Also, much research has been done in these papers that provides a rigorous treatment of the problem of regularization. In this paper we consider a regularization method for Feynman integrals which

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may be treated by the method of real variables, introducing in the process a regularization parameter δ. The regularization method may be unambiguously applied to *multiple* Feynman integrals in *Minkowski* space, and provides a method for rigorous treatment of Feynman integrals. This is the main purpose of the present work, as it allows us to change orders of integrations at will, and obtain existence criteria as a function of the regularization parameter δ. A regularization scheme is not useful unless it may be unambiguously and uniquely applied to multiple integrals and convergence of the regularized integrals may be established not only in Euclidean space, as is usually done, but also in Minkowski space. We develop a power counting theorem for the regularized Feynman integrals defined in *momentum* space, and a power counting convergence criterion which puts a restriction on the parameter δ as a function of the so-called power asymptotic coefficients of Feynman integrands. The regularization method is so chosen that we are able to bound the absolute value of Feynman integrals in Minkowski space by their corresponding Euclidean integrals and then study the resulting integrals as a function of the parameter δ. The regularization method is described in the next section, and the power counting convergence criterion is studied in Section 3. Finally we carry out an application to quantum electrodynamics in Section 4.

2. THE REGULARIZATION PROCEDURE

A Feynman integral may be written, in momentum space, in the by-now familiar form

$$F(P,\mu,\varepsilon)=\int_{\mathbb{R}^{4n}} dK A(P,K,\mu,\varepsilon)\prod_{l=1}^{L} D_l^{-1} \tag{1}$$

$$dK=\prod_{i=1}^{n}(dk_i)=\prod_{i=1}^{n}\prod_{j=0}^{3} dk_i^j \tag{2}$$

$$K=\left(k_1^0,\dots,k_n^3\right),\qquad P=\left(p_1^0,\dots,p_m^3\right),\qquad \mu=\left(\mu^1,\dots,\mu^\rho\right) \tag{3}$$

$$D_l=\left[Q_l^2+\mu_l^2-i\varepsilon\left(\mathbf{Q}_l^2+\mu_l^2\right)\right] \tag{4}$$

$$Q_l=\sum_{j=1}^{n}\alpha_{lj}k_j+\sum_{j=1}^{m}b_{lj}p_j \tag{5}$$

We consider only the case with $\mu_l > 0$. We define a regularized Feynman integral by making the change

$$\prod_{i=1}^{n}(dk_i) \to \prod_{i=1}^{n}\left(\frac{m^2}{k_i^2+m^2-i\varepsilon(\mathbf{k}_i^2+m^2)}\right)^{2\delta}(dk_i), \qquad \delta > 0 \tag{6}$$

where m is an arbitrary mass scale. The choice of regularization in (6) is quite convenient as we may bound the integral

$$F_\delta(P,\mu,\varepsilon) = \int \prod_{i=1}^{n}\left(\frac{m^2}{k_i^2+m^2-i\varepsilon(\mathbf{k}_i^2+m^2)}\right)^{2\delta} \times (dk_i)A(P,K,\mu,\varepsilon)\prod_{l=1}^{L}D_l^{-1} \tag{7}$$

in absolute value by

$$C_{\varepsilon,\delta}\int \prod_{i=1}^{n}\left(\frac{m^2}{k_{iE}^2+m^2}\right)^{2\delta}(dk_i)|A(P,K,\mu,\varepsilon)|\prod_{l=1}^{L}D_{lE}^{-1} \tag{8}$$

where

$$C_{\varepsilon,\delta} = \left[\frac{1}{\varepsilon}+\left(1+\frac{1}{\varepsilon^2}\right)^{1/2}\right]^{2\delta+L}$$

$$k_{iE}^2 = (k_i^0)^2+\mathbf{k}_i^2, \qquad D_{lE} = Q_{lE}^2+\mu_l^2 \tag{9}$$

In writing the inequality leading to (8), we used the bound (Zimmermann, 1968; Hahn and Zimmermann, 1968)

$$\left|[Q^2+\mu^2-i\varepsilon(\mathbf{Q}^2+\mu^2)]^{-1}\right| \leqslant (Q_E^2+\mu^2)^{-1}\left[\frac{1}{\varepsilon}+\left(1+\frac{1}{\varepsilon^2}\right)^{1/2}\right] \tag{10}$$

For $\varepsilon > 0$, it is then sufficient to study the absolute convergence of the *Euclidean* counterpart of (7) defined by

$$F_\delta(P,\mu,\varepsilon)_E = \int \prod_{i=1}^{n}\left(\frac{m^2}{k_{iE}^2+m^2}\right)^{2\delta}(dk_i)A(P,K,\mu,\varepsilon)\prod_{l=1}^{L}D_{lE}^{-1} \tag{11}$$

3. POWER COUNTING CRITERION

We develop a power counting convergence criterion for the integral in (11) (with $\varepsilon \geqslant 0$) in terms of the parameter δ. Since the analysis given below is based on an induction argument over the dimensionality $4n$ of the integral in (11), we may further bound the integral in (11) in a form suitable for carrying out the analysis over *one*-dimensional subspaces first. To this end note that

$$\left(k_{iE}^2+m^2\right)^{-1} \leqslant \prod_{j=0}^{3}\left[\left(k_i^j\right)^2+m^2\right]^{-1/4} \tag{12}$$

Hence we are led to consider integrals of the form

$$\int \prod_{i=1}^{4n}\left(1+x_i^2\right)^{-\delta/2} dx_i\, f(x_1, x_2, \ldots) \tag{13}$$

for $\delta > 0$. We make use of the fact that Feynman integrands belong (Weinberg, 1960; Fink, 1968; Manoukian, 1982a) to class B_{4n+4m}. That is, let $\mathbf{P}$ be a vector in the $(4n+4m)$-dimensional Euclidean space $\mathbb{R}^{4n+4m}$, such that $k_1^0, \ldots, k_n^3, p_1^0, \ldots, p_m^0$ may be written as linear combinations of the components of the vector $\mathbf{P}$. Suppose

$$\mathbf{P} = \mathbf{L}_1\eta_1 \cdots \eta_k + \cdots + \mathbf{L}_k\eta_k + \mathbf{C} \tag{14}$$

$1 \leqslant k \leqslant 4n+4m$, where $\mathbf{L}_1, \ldots, \mathbf{L}_k$ are independent vectors. Then we may find constants $b_1 > 1, \ldots, b_k > 1$ such that for $\eta_1 \geqslant b_1, \ldots, \eta_k \geqslant b_k$, we may write for the integrand $A(P, K, \mu, \varepsilon)\prod_{l=1}^{L} D_{lE}^{-1} \equiv \mathscr{E}(\mathbf{P})$:

$$\mathscr{E}(\mathbf{P}) = O\left(\eta_1^{\alpha(\{\mathbf{L}_1\})} \cdots \eta_k^{\alpha(\{\mathbf{L}_1, \ldots, L_k\})}\right) \tag{15}$$

where the $\alpha(\{\mathbf{L}_1, \ldots, \mathbf{L}_i\})$ are real numbers called power asymptotic coefficients, and depend on the subspace $\{\mathbf{L}_1, \ldots, \mathbf{L}_i\}$ generated by the vectors $\mathbf{L}_1, \ldots, \mathbf{L}_i$, $i = 1, \ldots, k$.

Consider the one-dimensional integral

$$F(\mathbf{P}) = \int_{-\infty}^{\infty} (1+x^2)^{-\delta/2} dx f(\mathbf{P}+\mathbf{L}x) \tag{16}$$

where f belongs to class B_{4n+4m}. By the Heine–Borel covering theorem we may write (Weinberg, 1960) the interval $(-\infty, \infty)$ as the union of the following sets:

$$J^{\pm} = \{x : x = z\eta_1 \cdots \eta_k,\ |z| = \pm z \geqslant b_0 > 1\} \tag{17}$$

$$J^{\pm}_{i_1 \cdots i_r} = \{x : x = U_{i_1}\eta_1 \cdots \eta_k + \cdots + U_{i_1 \cdots i_r}\eta_r \cdots \eta_k + z\eta_{r+1} \cdots \eta_k$$

$$b_0(i_1 \cdots i_r) \leqslant |z| = \pm z \leqslant \eta_r \lambda_{i_1 \cdots i_r}\}, \qquad 1 \leqslant r \leqslant k \tag{18}$$

$$J^{0\pm}_{i_1 \cdots i_k} = \{x : x = U_{i_1}\eta_1 \cdots \eta_k + \cdots + U_{i_1 \cdots i_k}\eta_k + z$$

$$0 \leqslant |z| = \pm z \leqslant b_0(i_1 \cdots i_k)\} \tag{19}$$

where $\mathbf{P} = \mathbf{L}_1\eta_1 \cdots \eta_k + \cdots + \mathbf{L}_k\eta_k + \mathbf{C}$. We may then bound the integral in (16) as

$$|F(\mathbf{P})| \leqslant \sum_{\pm} \int_{J^{\pm}} |x|^{-\delta} dx\, |f(\mathbf{P} + \mathbf{L}x)| + \sum_{\pm} \sum_{r=1}^{k} \sum_{i_1 \cdots i_r} \int_{J^{\pm}_{i_1 \cdots i_r}} dx\, |f(\mathbf{P} + \mathbf{L}x)|$$

$$+ \sum_{\pm} \sum_{i_1 \cdots i_k} \int_{J^{0\pm}_{i_1 \cdots i_k}} dx\, |f(\mathbf{P} + \mathbf{L}x)| \tag{20}$$

A bound to the integrals $\int_{J^{\pm}}$ may be readily obtained to be given by

$$M \int_{b_0}^{\infty} d|z|\, |z|^{-\delta + \alpha(\{\mathbf{L}\})} \eta_1^{\alpha(\{\mathbf{L}, \mathbf{L}_1\})} \cdots \eta_k^{\alpha(\{\mathbf{L}, \mathbf{L}_1, \ldots, \mathbf{L}_k\})} \eta_1 \cdots \eta_k \tag{21}$$

for some constants $M > 0, b_1 > 1, \ldots, b_k > 1$, with $\eta_1 \geqslant b_1, \ldots, \eta_k \geqslant b_k$, and the latter integral exists for

$$\delta > 1 + \alpha(\{\mathbf{L}\}) \tag{22}$$

If we denote the subspace $\{\mathbf{L}\}$ by I, then for any $\varepsilon' > 0$, the condition

$$\delta > 1 + \alpha(\{\mathbf{L}\}) + \varepsilon' = \max_{S' \subset I} [\alpha(S') + \dim S'] + \varepsilon' \tag{23}$$

implies the convergence of the integral (21) and hence also the integrals $\int_{J^{\pm}}$ in (20). (Note that, trivially, I is the only non-null one-dimensional subspace of the one-dimensional space I. Also note that for $x \in J^{\pm}$, $x = z\eta_1 \cdots \eta_k$.) It is interesting to note that with the criterion in (23) satisfied, we may bound the expression in (21) by

$$M'\frac{1}{\left(\delta-\max\limits_{S'\subset I}\left[\alpha(S')+\dim S'\right]-\varepsilon'\right)}\eta_1^{\alpha(\{\mathbf{L},\mathbf{L}_1\})+1}\cdots\eta_k^{\alpha(\{\mathbf{L},\mathbf{L}_1,\ldots,\mathbf{L}_k\})+1} \tag{24}$$

thus exhibiting a singularity for $\delta\to+(\max[\alpha(S')+\dim S']+\varepsilon')$. The introduction of the arbitrary parameter $\varepsilon'>0$ will be convenient for generalizations to arbitrary dimensions. Upper-bound values for the remaining integrals in (20) may be readily obtained, by using in the process that for any $\varepsilon'>0$, $\ln\eta\leqslant\eta^{\varepsilon'}/\varepsilon'$, for $\eta\geqslant 1$, to obtain with the convergence criterion in (23) satisfied:

$$|F(\mathbf{P})|\leqslant C\eta_1^{\alpha_I(\{\mathbf{L}_1\})}\cdots\eta_k^{\alpha_I(\{\mathbf{L}_1,\ldots,\mathbf{L}_k\})} \tag{25}$$

where, in a standard notation,

$$\alpha_I(\{\mathbf{L}_1,\ldots,\mathbf{L}_r\})=\max_{\Lambda(I)S'=S_r}\left[\alpha(S')+\dim S'-\dim S_r\right]+\varepsilon' \tag{26}$$

$S_r\equiv\{\mathbf{L}_1,\ldots,\mathbf{L}_r\}$, with ε' an arbitrary positive number that may be chosen as small as we please. Also, there exist constants $b_1'>1,\ldots,b_k'>1$ such that $\eta_1\geqslant b_1',\ldots,\eta_k\geqslant b_k'$ in (25). (Note that with the restriction $\varepsilon'>0$, *no* logarithmic growth occurs in the analysis. This condition will be sufficient for our purposes in this work.)

Now we generalize our results to arbitrary dimensions by induction. To this end we consider the integral

$$F(\mathbf{P})=\int_{-\infty}^{\infty}\left(1+x_1^2\right)^{-\delta/2}dx_1$$
$$\times\prod_{i=2}^{4n}\int_{-\infty}^{\infty}\left(1+x_i^2\right)^{-\delta/2}dx_i\,f(\mathbf{P}+\mathbf{L}_1x_1+\cdots+\mathbf{L}_{4n}x_{4n}) \tag{27}$$

where $\mathbf{L}_1,\ldots,\mathbf{L}_{4n})$ is an orthonormal set of vectors. Let I_2 be a $(4n-1)$-dimensional subspace with which the integral $\int\prod_{i=2}^{4n}dx_i(\cdot)$ is associated, and let the subspace with which the one-dimensional x_1 integration is associated be denoted by I_1. As induction hypotheses suppose that the I_2 integral belongs to class B_K $(K=4m+1)$ and is absolutely convergent for

$$\delta>\max_{S'\subset I_2}\left[\alpha(S')+\dim S'\right]+\varepsilon_2 \tag{28}$$

where ε_2 is an arbitrary small positive number, with asymptotic coefficients of the standard form

$$\alpha_{I_2}(S) = \max_{\Lambda(I_2)S'=S} \left[\alpha(S')+\dim S' - \dim S\right] + \varepsilon_2 \tag{29}$$

Again with the restriction $\varepsilon_2 > 0$ no logarithmic growth occurs in the analysis. From the one dimensional analysis, given above, we then conclude that (27) is absolutely convergent if

$$\delta > \max_{S'' \subset I_1} \left[\alpha_{I_2}(S'')+\dim S''\right)\right] + \varepsilon_1 \tag{30}$$

for an arbitrary small positive number ε_1. From (29) we may rewrite (Weinberg, 1960) the condition in (30) as

$$\begin{aligned} \delta &> \max_{S \subset I} \left[\alpha(S)+\dim S\right] + \varepsilon_1 + \varepsilon_2 \\ &\equiv \max_{S \subset I} \left[\alpha(S)+\dim S\right] + \varepsilon' \end{aligned} \tag{31}$$

Since $\max_{S \subset I}[\alpha(S)+\dim S] \geqslant \max_{S \subset I_2}[\alpha(S)+\dim S]$, we may then state, by an application of Fubini's theorem in the process, that (27) is absolutely convergent if the power counting criterion in (31) is satisfied, and its value is independent of the order of integrations taken. The asymptotic coefficients of $F(\mathbf{P})$ may be readily obtained from (26) and (29) to be given by

$$\alpha_I(S) = \max_{\Lambda(I)S'=S} \left[\alpha(S')+\dim S' - \dim S\right] + \varepsilon' \tag{32}$$

We may then state that the integral in (11) (and (7) for $\varepsilon > 0$) is absolutely convergent for $\delta > \max_{S \subset I}[\alpha(S)+\dim S]+\varepsilon'$, where ε' is an arbitrary positive number, I is the subspace associated with the integration variables, and the $\alpha(S)$ are the power asymptotic coefficients of the Feynman integrand. The distributional $\varepsilon \to +0$ limit of the integral in (7) may be carried out as in (Lowenstein and Speer, 1976; Manoukian, 1983a).

The power counting convergence criterion applies to Feynman integrals even before subtractions are carried out as long as the convergence criterion in (31) is satisfied. For subtracted integrands

$$\max_{S \subset I} \left[\alpha(S)+\dim S\right] \leqslant -1$$

(Manoukian, 1982b), and since ε' is an arbitrary positive small number which may be taken to be smaller than one, we note from (31) that δ may be taken to be zero as expected.

4. APPLICATION TO QED

Recently (Manoukian, 1983b) the so-called Schwinger's (Schwinger, 1953, 1951; Johnson, 1965) gauge-invariant procedure for the current by a suitable limiting method of a line integral has been applied to the whole expression for the vacuum-to-vacuum transition amplitude for QED in the presence of external source, thus taking into consideration all closed fermion loops. Our method of regularization in this paper to QED consists of using this gauge-invariant procedure, *then* apply the loop integration regularization developed in Section 3. The vacuum-to-vacuum transition amplitude in the presence of external sources, taking into account Schwinger's line integral for the current, is

$$\langle 0_+|0_-\rangle = N\int [DA]\exp\left[i\int (dx)(dx')\bar{\eta}(x)G(x,x';eA)\eta(x')\right]$$

$$\times\exp\left\{-\sum_{N=2,4,6,\dots}\frac{(e)^N}{N}\int\frac{(dQ_1)}{(2\pi)^2}\cdots\frac{(dQ_N)}{(2\pi)^2}\right.$$

$$\times\delta(Q_1+\cdots+Q_N)\,A^{\mu_1}(Q_1)\dots A^{\mu_N}(Q_N)$$

$$\left.\times\,\pi_{\mu_1\dots\mu_N}(Q_1,\dots,Q_N)\right\}\exp\left[i\int\mathscr{L}(A,J)\right] \tag{33}$$

where $G(x,x';eA)$ satisfies the differential equation in the presence of the "classical" vector potential $A^\mu(x)$:

$$\left[\frac{\gamma\partial}{i}+m-e\gamma_\mu A^\mu(x)\right]G(x,x';eA)=\delta(x-x') \tag{34}$$

Also

$$\pi_{\mu_1\dots\mu_N}(Q_1,\dots,Q_N)=\int\frac{(dp)}{(2\pi)^4}\theta_{(N)}I_{\mu_1\dots\mu_N}(Q_1,\dots,Q_N;p) \tag{35}$$

where

$$\theta_{(N)}=1+\sum_{j=1}^{N}\sum_{1\leqslant i_1<\cdots<i_j\leqslant N}(-1)^j\theta_{i_1}\dots\theta_{i_j} \tag{36}$$

and θ_i is the operation of setting $Q_i=0$. $\mathscr{L}(A,J)$ is the Lagrangian density of the photon field in the presence of an external current $J_\mu(x)$. N is a normalization factor. All the Green's functions are obtained from (32) by functional differentiations with respect to the external sources $\eta(x)$, $\bar{\eta}(x)$ and $J_\mu(x)$. The objects $I_{\mu_1 \ldots \mu_N}(Q_1, \ldots, Q_N; p)$ are given by

$$I_{\mu_1 \ldots \mu_N}(Q_1, \ldots, Q_N; p)$$
$$= \frac{1}{N!} \sum_{[i_1 \ldots i_N]} \operatorname{Tr}\left[\gamma_{\mu_{i_1}} S(p) \gamma_{\mu_{i_2}} S(p - Q_{i_2}) \ldots \gamma_{\mu_{i_N}} S(p - Q_{i_2} - \cdots - Q_{i_N})\right] \tag{37}$$

for $N = 6, 8, 10, \ldots$, and $\Sigma_{[i_1 \ldots i_N]}$ denotes a summation over all permutations of the indices in $(1, \ldots, N)$;

$$I_{\mu_1 \ldots \mu_4}(Q_1, \ldots, Q_4; p) = \frac{1}{4!} \sum_{i=1}^{4} \sum_{[i_1 i_2 i_3]'}^{(i)} \{\cdot\} \tag{38}$$

$$\{\cdot\} = \operatorname{Tr}\left[\gamma_{\mu_i} S(p) \gamma_{\mu_{i_1}} S(p - Q_{i_1}) \ldots \gamma_{\mu_{i_3}} S(p - Q_{i_1} - Q_{i_2} - Q_{i_3})\right]$$
$$+ \frac{1}{3!} \frac{\partial}{\partial p^{\mu_{i_1}}} \frac{\partial}{\partial p^{\mu_{i_2}}} \frac{\partial}{\partial p^{\mu_{i_3}}} \operatorname{Tr}\left[\gamma_{\mu_i} S(p)\right] \tag{39}$$

and $\Sigma^{(i)}_{[i_1 i_2 i_3]'}$ denotes a summation over all permutations of the indices in $(1, \ldots, \hat{i}, \ldots, 4)$ with the index i omitted in the latter. Finally we have the well known expression for $I_{\mu_1 \mu_2}(Q_1, Q_2; p)$:

$$I_{\mu_1 \mu_2}(Q_1, Q_2; p) = \operatorname{Tr}\left[\gamma_{\mu_1} S\left(p + \frac{Q_2}{2}\right) \gamma_{\mu_2} S\left(p - \frac{Q_2}{2}\right)\right]$$
$$+ \left[1 + \frac{1}{24}\left(Q_1 \frac{\partial}{\partial p}\right)^2\right] \frac{\partial}{\partial p^{\mu_1}} \operatorname{Tr}\left[\gamma_{\mu_2} S(p)\right] \tag{40}$$

We may now apply the $i\varepsilon$ prescription in (4), and apply our regularization procedure by the substitution of the measures of integrations over the internal loops in (33) as

$$(dQ_i) \to (dQ_i)\left(\frac{m^2}{Q_i^2 + m^2 - i\varepsilon(\mathbf{Q}^2 + m^2)}\right)^{2\delta}, \qquad i = 1, 2, \ldots, N \tag{41}$$

Having done this, we obtain a regularization of a fully gauge-invariant procedure for QED through the vacuum-to-vacuum transition amplitude yielding regularized Feynman rules. It is interesting to apply the regularization procedure to compute, for example, the self-mass of the electron to lowest order in α, as a function of the regularization parameter δ for $\delta \sim 0$. To this end the proper self-energy part of the electron propagator (cf., Jauch and Rohrlich, 1976) is given in a Euclidean metric to be:

$$\sum_{\delta}(p) = -\frac{2ie^2}{(2\pi)^4}\int \frac{(dk_E)}{k^2}\frac{\gamma(p-k)+2M}{\left[(p-k)^2+M^2\right]}\left(\frac{m^2}{k^2+m^2}\right)^{2\delta} \tag{42}$$

Upon writing

$$\frac{1}{(p-k)^2+M^2} = \frac{1}{(p-k)^2} - \frac{M^2}{(p-k)^2\left[(p-k)^2+M^2\right]} \tag{43}$$

$$\left(\frac{m^2}{k^2+m^2}\right)^{2\delta} = \left(\frac{m^2}{k^2}\right)^{2\delta} + (m^2)^{2\delta}\left[\frac{1}{(k^2+m^2)^{2\delta}} - \frac{1}{(k^2)^{2\delta}}\right] \tag{44}$$

and using the four-dimensional averages:

$$\left\langle \frac{1}{(p-k)^2} \right\rangle = \frac{1}{p_>^2}, \qquad \left\langle \frac{pk}{(p-k)^2} \right\rangle = \frac{|p||k|}{2p_>^2}\left(\frac{p_<}{p_>}\right) \tag{45}$$

in a standard notation, we obtain for δM $(\alpha = e^2/4\pi)$

$$\delta M = \frac{3\alpha}{4\pi}M\left(\frac{1}{2\delta}\right) + Mc_1 \tag{46}$$

where c_1 is independent of the parameter δ in the limit $\delta \to 0$. The factor $(1/2\delta)$ replaces the familiar logarithmic divergence term in δM. Our procedure is different from the dimensional regularization one and proceeds by adopting the regularization in (41) in general, and, for QED, by developing first a gauge-invariant procedure for $\langle 0_+|0_-\rangle$, through the tensors $\pi_{\mu_1 \dots \mu_N}$ in (35), by Schwinger's intuitive appealing method, and then applying the regularization in (41). The similar study for non-Abelian gauge theories is much more difficult and will be discussed in a subsequent report; the role of the parameter δ in the language of renormalization group methods will be also analyzed.

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